ON ASSOCIATIVE CONFORMAL ALGEBRAS OF LINEAR GROWTH

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Abstract. We introduce the notions of conformal identity and unital associative conformal algebras and classify finitely generated simple unital associative conformal algebras of linear growth. These are precisely the complete algebras of conformal endomorphisms of finite modules.

0. Introduction.

The subject of conformal algebras is a relatively recent development in the theory of vertex algebras [K1]. The relation of Lie conformal algebras to vertex algebras is similar to that of Lie algebras and their universal enveloping algebras.

Semisimple Lie conformal algebras of finite type were classified in [DK] and semisimple associative algebras of finite type in [K2]. Associative conformal algebras appear in conformal representation theory. The complete algebras of conformal endomorphisms of finite modules CEnd_N (called in this paper the conformal Weyl algebras) are a particular example(see [K1, 2.9]). Notice that these are algebras are not finite as modules over $\mathbb{C}[\partial]$.

These paper is concerned with associative algebras of finite growth (but not necessarily of finite type). We require the conformal algebra to be unital, that is, to possess an element that acts as a left identity with respect to the 0-th multiplication and whose locality degree with itself is 1. In particular, this means that its 0-th coefficient is the (left) identity of the coefficient algebra. One can then use the classification of associative algebras of linear growth obtained in [SSW] to classify a class of unital conformal algebras.

Main Theorem. *Let C be a finitely generated simple unital associative conformal algebra. If C has growth 1, then it is a conformal Weyl algebra $CEnd_N$.*

Notice that there is a marked difference from the case of growth 0 where simple associative conformal algebras are just the current algebras over the matrix algebras ([K2, 4.4], see also Remark 4.3 below).

The paper is organized as follows. The first chapter is devoted to preliminary material on conformal algebras where, for the most part, we loosely follow the treatment of [K2]. The second chapter discusses the concept of Gelfand-Kirillov dimension for conformal algerbas and relates it to the dimension of the coefficient algebra. The third chapter is devoted to the discussion of unital conformal algebras. Here the main result (Proposition 3.5) relates the presence of conformal identity to the structure of the coefficient algebra. The fourth chapter contains the proof of the Main Theorem in Theorem 4.6. We also classify unital associative algebras of growth 0 in Theorem 4.2 (the proof is different from [K2] which contains the general case).

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1. Preliminaries on Conformal Algebras.

We start with a motivation for the concept of the conformal algebra. Let R be a (Lie or associative) algebra. One can consider formal distributions over R, that is elements of $R[[z, z^{-1}]]$. These appear, for instance, in the theory of operator product expansions in vertex algebras. Obviously, it is impossible to multiply two formal distributions; however, one can introduce an infinite number of bilinear operations that act as a "replacement" for multiplication. Let $a(z), b(z)$ be two formal distributions. For an integer $n \geq 0$ define another formal distribution

$$a(z) \, (n) \, b(z) = \mathrm{Res}|_{z=0} \, a(z)b(w)(z-w)^n \tag{1.1}$$

called the n-th product of $a(z)$ and $b(z)$. (Those familiar with vertex algebras will observe that we do not consider the -1st, i.e. the Wick, product here as its definition requires a representation of R.) If one writes $a(z)$ explicitly as $\sum a(n)z^{-n-1}$, it follows that

$$(a(z) \, (n) \, b(z))(k) = \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} a(n-j)b(k+j). \tag{1.2}$$

In the theory of vertex algebras one wants to consider only the mutually local series, i.e. $a(z)$ and $b(z)$ such that $a(z) \, (n) \, b(z)$ and $b(z) \, (n) \, a(z)$ are zero for $n >> 0$. We simply say that $a(z)$ and $b(z)$ are local if $a(z) \, (n) \, b(z) = 0$ for $n > N(a, b)$ and call minimal such $N(a, b)$ the degree of locality of $a(z)$ and $b(z)$. In terms of coefficients this becomes:

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} a(n-j)b(k+j) = 0, \quad n >> 0, k \in \mathbb{Z}. \tag{1.3}$$

It is possible to rewrite this statement as follows:

Lemma 1.1. [K1, 2.3] *The series $a(z)$ and $b(z)$ are local iff $a(z)b(w)(z-w)^n = 0$ for $n > N(a,b)$. In terms of coefficients:*

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}a(l-j)b(m+j) = 0, \quad n > N(a,b), l, m \in \mathbb{Z}. \tag{1.4}$$

Apart from locality one also wants to take into account the action of $\partial = \partial/\partial z$. These consideration suggested the following definition, first stated in [K1].

Definition 1.2. A conformal algebra C is a $\mathbb{C}[\partial]$-module endowed with bilinear products (n), $n \in \mathbb{Z}_{\geq 0}$, that satisfy the following axioms for any $a, b \in C$:

(C1) (locality) $a \, (n) \, b = 0$ for $n > N(a,b)$;

(C2) (Leibnitz rule) $\partial(a \, (n) \, b) = (\partial a) \, (n) \, b + a \, (n) \, (\partial b)$;

(C3) $(\partial a) \, (n) \, b = -na \, (n-1) \, b$.

Clearly algebras of formal distributions closed with respect to the action of ∂ satisfy these axioms.

The bilinear product (n) is usually called the multiplication of order n.

In general, $N(a,b) \neq N(b,a)$. Moreover, even if a, b and c are pairwise local, $a \, (n) \, b$ and c need not be such. One can, however, establish some correlations between different degrees of locality. For instance, it follows from (C3) that $N(\partial a, b) = N(a,b) + 1$ and from (C2) and (C3) that $N(a, \partial b) = N(a,b) + 1$. It follows as well that $(\partial a) \, (0) \, b = 0$ for all a, b. Yet, one can hardly make deeper statements about conformal algebras without restricting attention to less general cases.

Two such cases are algebras of formal distributions over Lie and associative algebras closed with respect to the derivation ∂. Here taking products does not destroy locality:

Lemma 1.3. (Dong's lemma) [Li, K1] *Let a, b and c be pairwise mutually local formal distributions over either a Lie or associative algebra. Then for any $n \geq 0$ $a \, (n) \, b$ and c are again pairwise mutually local.*

We remarked above that ∂ preserves locality (although not its degree) as well. Thus for such algebras of formal distributions it is enough to check locality of the generators. Examples follow:

Example 1.4. 1. *Current algebras*: let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}[t, t^{-1}]$ its affine extension; $[gt^m, ht^m] = [g,h]t^{m+n}$. The algebra of formal distributions generated by $g(t) = \sum gt^n z^{-n-1}$ and their derivatives is a conformal algebra. The non-zero products for generators are

$$g(t) \, (0) \, h(t) = [g,h](t).$$

This algebra is denoted $\mathrm{Cur}(\mathfrak{g})$.

2. *(conformal) Virasoro algebra*: let t, t^{-1}, ∂_t be the generators of the Lie algebra of differential operators on S^1 (i.e. a centerless Virasoro algebra). Consider the conformal algebra generated by the formal distribution $L = -\sum \partial_t t^n z^{-n-1}$ and its derivatives. (This is a non-standard way of writing the generator of the conformal Virasoro algebra and, for that matter, of the centerless Virasoro algebra. We put powers of t on the left since later it will become the generating indeterminate of a polynomial extension; see below.) The non-zero products for the generator are

$$L \textcircled{0} L = \partial L \text{ and } L \textcircled{1} L = 2L.$$

3. *Associative current algebras*: let A be an associative algebra and $A[t, t^{-1}]$ its affine extension (t commutes with A). The algebra of formal distributions generated by $f_a = \sum a t^n z^{-n-1}$ and its derivatives is a conformal algebra. The non-zero products for the generators are

$$f_a \textcircled{0} f_b = f_{ab}.$$

This algebra is denoted $\mathrm{Cur}(A)$.

4. *(conformal) Weyl algebra* $\mathfrak{W}$: let t, ∂_t be the generators of the Weyl algebra W with the relation $\partial_t t - t\partial_t = 1$. Consider its localization at t denoted here by W_t. Formal distributions $e = \sum t^n z^{-n-1}$ and $L = \sum \partial_t t^n z^{-n-1}$ with their derivatives generate a conformal algebra. The non-zero products for the generators are

$$\begin{aligned}
&e \textcircled{0} e = e, e \textcircled{0} L = L \textcircled{0} e = L,\\
&e \textcircled{1} L = L \textcircled{1} e = -e,\\
&L \textcircled{0} L = \partial_t L, L \textcircled{1} L = -L.
\end{aligned}$$

The algebra of formal distributions over $\mathcal{M}\mathrm{at}_n(W_t)$, which is a generalization of the above example, will also be called a (conformal) Weyl algebra.

Examples 1 and 2 were studied extensively in [DK]. Obviously, the third example mimics example 1. The Weyl algebra and the Virasoro algebra are also closely related. Since any associative conformal algebra can be turned into a Lie conformal algebra ([K1, 2.10a]), the Weyl algebra will contain a subalgebra isomorphic to the Virasoro algebra (and e will become a central element). Again, let us remark that the generator of the Virasoro algebra is usually written as $L = -\sum t^n \partial_t z^{-n-1}$ (the results of conformal multiplications are the same here). In a similar manner, one can choose the formal distribution $\sum t^n \partial_t z^{-n-1} = L + \partial e$ as a generator of the Weil algebra instead of L.

Another approach to the Weyl algebra (or rather the matrix algebra over the Weyl algebra) is to consider the complete algebra of conformal endomorphisms of a finite $\mathbb{C}[\partial]$-module. For a discussion of conformal representation theory see [DK] and [K2].

It is possible to generalize example 4 by considering a conformal algerba of series in $A[t,t^{-1};\delta][[z,z^{-1}]]$ where A is an associative algebra, t an indeterminate, and δ a locally nilpotent derivation of A. Such algebras are called *differential algebras.* We postpone their discussion until Chapter 3.

One obviously wants to formalize the above discription of Lie and associative algebras (and, in particular, Dong's lemma 1.3) of formal distributions to the general case of conformal algebra.

Remark, first, that the 0-th product of a Lie or an associative algebra of formal distributions behaves just as a regular Lie or associative product. It is certainly possible to forget about products of positive order and consider the algebra $(C, \, ⓪ \,)$. However, one wants to take into account the full structure of the conformal algebra, not just its 0-th product.

The following construction was first partially introduced in [Bo] for vertex algebras and generalized in [K1] for conformal algebras.

Notice first that by axioms (C2) and (C3), ∂C is an ideal in $(C, \, ⓪ \,)$. Consider now a conformal algebra $\tilde{C} = C[t,t^{-1}]$. Define the n-th product as

$$at^l \, (n) \, bt^m = \sum_{j\in\mathbb{Z}_+} \binom{l}{j} (a \, (n+j) \, b) t^{l+m-j}. \tag{1.5}$$

and put $\tilde{\partial} = \partial + \partial/\partial t$. This makes $\tilde{C}$ into a conformal algebra.

Definition 1.5. ([K1]) The coefficient algebra $\operatorname{Coeff} C$ of conformal algebra C is the quotient algebra $(\tilde{C}, \, ⓪ \,)/\tilde{\partial}\tilde{C}$.

Denote the map $\tilde{C} \to \operatorname{Coeff} C$ by ϕ. Consider the algebra of formal distributions over $\operatorname{Coeff} C$ consisting of series $\sum \phi(at^n) z^{-n-1}$ where $a \in C$. This algebra is isomorphic to C. It obviosuly follows that every conformal algebra can be written as an algebra of formal distributions over its coefficient algebra.

In the remaining chapters we will often write $\phi(at^n)$ as simply $a(n)$, i.e. we will not distinguish between a conformal algebra and the corresponding algebra of formal distributions. The following formula will be quite useful:

$$(\partial a)(n) = -na(n-1), \quad n \in \mathbb{Z}. \tag{1.6}$$

We say that C is conformal Lie or associative if $\operatorname{Coeff} C$ is respectively Lie or associative (this definition can be applied to any variety of algebras as well). In

particular a conformal algebra is Lie conformal if

$$\begin{aligned} f \,\textcircled{n}\, g &= \sum_{j\in\mathbb{Z}_+} \frac{(-1)^{j+n}}{j!}\partial(g \,\textcircled{\scriptsize $n+j$}\, f), \\ f \,\textcircled{m}\, (g \,\textcircled{n}\, h) &= \sum_{j\in\mathbb{Z}_+} \binom{m}{j} (f \,\textcircled{j}\, g) \,\textcircled{\scriptsize $m+n-j$}\, h + g \,\textcircled{n}\, (f \,\textcircled{m}\, h) \end{aligned} \tag{1.7}$$

and associative conformal if

$$f \,\textcircled{m}\, (g \,\textcircled{n}\, h) = \sum_{j\in\mathbb{Z}_+} \binom{m}{j} (f \,\textcircled{j}\, g) \,\textcircled{\scriptsize $m+n-j$}\, h. \tag{1.8}$$

The latter condition is equivalent to

$$(f \,\textcircled{m}\, g) \,\textcircled{n}\, h = \sum_{j\in\mathbb{Z}_+} (-1)^j \binom{m}{j} f \,\textcircled{\scriptsize $m-j$}\, (g \,\textcircled{\scriptsize $n+j$}\, h). \tag{1.9}$$

Many standard definitions from the "classical" structure theory carry over to the conformal case in an obvious manner. Thus, a conformal ideal is a conformal subalgebra closed under all right and left multiplications of any order, a simple conformal algebra is an algebra that contains no ideals, and so on.

Below all conformal algebras are assumed to be associative conformal unless otherwise stated.

2. Gelfand-Kirillov Dimension of Conformal Algebras.

The Gelfand-Kirillov dimension of a finitely generated algebra (of any variety) A is defined as

$$\operatorname{GKdim} A = \limsup_{r\to\infty} \frac{\log \dim(V^1 + V^2 + \cdots + V^r)}{\log r},$$

where V is a generating subspace of A. This definition easily carries over to the conformal case.

Let C be a finitely generated conformal algebra (over any variety) with generators $f_1, \dots, f_n$. Define C_r to be a $\mathbb{C}[\partial]$-span of products of less than r generators with any positioning of brackets and any orders of multiplication.

Since the powers of ∂ can be gathered in the beginning of conformal monomials (with a probable change in the orders of multiplications), it is clear that $\bigcup_r C_r = C$. For a given ordered collection of generators and a given positioning of brackets, the number of non-zero monomials is finite because of locality (C1). Therefore, $\operatorname{rk} C_r$ is finite.

Definition 2.1. The Gelfand-Kirillov dimension of a finitely generated conformal algebra C is

$$\operatorname{GKdim} C = \limsup_{r\to\infty} \frac{\log \operatorname{rk}_{\mathbb{C}[\partial]} C_r}{\log r}.$$

Moreover, when C is an associative conformal algebra, every element can be written as a sum of $(\dots(f_{j_1} \, (n_1) \, f_{j_2}) \, (n_2) \, f_{j_3} \dots) \, (n_{r-1}) \, f_{j_r}$ over $\mathbb{C}[\partial]$ with the rewriting process prescribed by (1.8) and (1.9) not increasing the number of generators involved in the original presentation of the given element. Therefore,

$$\begin{aligned} C_1 =& \operatorname{Span}_{\mathbb{C}[\partial]}(f_1, \dots, f_n), \\ C_r =& \operatorname{Span}_{\mathbb{C}[\partial]}(g \, (m) \, f_j \mid g \in C_{r-1}, m \geq 0, 1 \leq j \leq n) + C_{r-1}. \end{aligned} \tag{2.1}$$

Hereafter this description of C_r's will be used.

We will now show that in associative conformal algebras the orders of multiplications in the monomials used in the presentation (2.1) are uniformly bounded. The following lemma is a well-known fact (actually, it can be deduced directly from the standard proof of Dong's lemma 1.3); the proof is provided only to demonstrate the employment of rules (1.8) and (1.9).

Lemma 2.2. *Let $N = \max_{i,k} N(f_{j_i}, f_{j_k})$. If $(\dots(f_{j_1} \, (n_1) \, f_{j_2}) \, (n_2) \, f_{j_3} \dots) \, (n_{i-1}) \, f_{j_i} \neq 0$, then $n_j \leq N$ for all j.*

Proof. Denote $(\dots(f_{j_1} \, (n_1) \, f_{j_2}) \, (n_2) \, f_{j_3} \dots) \, (n_{i-3}) \, f_{j_{i-2}}$ by g. If $n_{i-1} > N$, then

$$\begin{aligned} (g \, (n_{i-2}) \, f_{j_{i-1}}) \, (n_{i-1}) \, f_{j_i} &= \sum_{s \geq n_{i-1}} \binom{n_{i-2}}{s - n_{i-1}} g \, (n_{i-2}+n_{i-1}-s) \, (f_{j_{i-1}} \, (s) \, f_{j_i}) \\ &= \sum_{s} \binom{n_{i-2}}{s - n_{i-1}} g \, (n_{i-2}+n_{i-1}-s) \, 0 = 0. \end{aligned}$$

The statement follows by induction. $\square$

One can also speak sometimes of the growth of C meaning the growth of function $\gamma_C(r) = \operatorname{rk}_{\mathbb{C}[\partial]} C_r$. The Gelfand-Kirillov dimension of C is finite if and only if $\gamma_C(r)$ is polynomial. Lemma 2.2 implies that $\operatorname{rk} C_r \leq \sum_{j=1}^{n} j^r N^{r-1}$. Hence, just as in the non-conformal case, $\gamma_C(r)$ can not be superexponential while exponential growth is possible (e.g. in free conformal algebras, see [Ro] for the definition and explicit construction of basis).

Remark 2.3. Similar results can be proven for Lie conformal algebras since for them the presentation (2.1) holds as well because of the Jacobi identity (1.7). However

the orders of multiplications in the Lie version of Lemma 2.2 will depend linearly on r ([K1], [Ro, 1.17].

The Gelfand-Kirillov dimension is invariant to the change of the generating set: the new generators are contained in some C_k which in turn is contained in some C'_l (here $C'_1, C'_2, \dots$ are the submodules defined by the new set of generators), thus for the new C'_r's, $C'_r \subseteq C_{kr} \subseteq C'_{lr}$ and the Gelfand-Kirillov dimension which measures only the growth of $\gamma_C(r)$ remains the same. Also, the Gelfand-Kirillov dimension of a subalgebra does not exceed the Gelfand-Kirillov dimension of the full algebra.

Just as in the case of non-finitely generated associative algebras, it is possible to define a Gelfand-Kirillov dimension of a non-finitely generated conformal algebra:

$$\operatorname{GKdim} C = \sup_{C' \subset C, C' \text{ finitely generated}} \operatorname{GKdim} C'.$$

If C is finite as a $\mathbb{C}[\partial]$-module, $\operatorname{GKdim} C = 0$. Conversely, if C is finitely generated as a conformal algebra and is an infinite $\mathbb{C}[\partial]$-module, it has a non-zero Gelfand-Kirillov dimension.

We will now relate the Gelfand-Kirillov dimension of C to that of $\operatorname{Coeff} C$. Recall that $\tilde{C} = C \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ and $\operatorname{Coeff} C = \tilde{C}/\tilde{\partial}\tilde{C}$ where $\tilde{\partial} = \partial + \partial/\partial t$. The map $(\tilde{C}, ⓪) \to \operatorname{Coeff} C$ is again denoted by ϕ.

Notice that by definition, $\tilde{\partial}(ft^k) = \partial f t^k + k f t^{k-1}$ where $\partial f t^k$ is a shorthand for $(\partial f)t^k$. Therefore, $\phi(\partial f t^k) = -k\phi(f t^{k-1})$. In general, if an element of $\operatorname{Coeff} C$ is an image of $\partial^i f t^k$, one can choose another preimage for it, $f' t^{k'}$, $f' \notin \partial C$. Remark also, that $\dim_{\mathbb{C}} \phi(C') \leq \operatorname{rk}_{\mathbb{C}[\partial]} C'$ for $C' \subseteq C$.

It is not difficult to see that passing from C to $\tilde{C}$ increases GKdim by 1. However, to figure out the effect of factoring out $\tilde{\partial}\tilde{C}$ and forgetting the contribution of all multiplications but ⓪ requires more work.

Theorem 2.4. *For an associative conformal algebra C,*
$GKdim\, Coeff\, C \leq GKdim\, C + 1$.

Proof. Notice that even when C is finitely generated, $\operatorname{Coeff} C$ does not have to be. Nonetheless, since we need to prove an upper bound on $\operatorname{GKdim} \operatorname{Coeff} C$, it suffices to demonstrate that such a bound holds for any finitely generated subalgebra of $\operatorname{Coeff} C$.

Let V be a generating subspace of $\operatorname{Coeff} C$. One can choose $f_1, \dots, f_n \in C$ and $M^-, M^+ \in \mathbb{Z}$ such that $V \subseteq \operatorname{Span}_{\mathbb{C}}(\phi(f_j t^k) \mid 1 \leq j \leq n, M^- \leq k \leq M^+)$. We can always assume that $M^- \leq 0$.

Let

$$\tilde{V} = \operatorname{Span}_{\mathbb{C}}(f_j t^k \mid 1 \leq j \leq n, M^- \leq k \leq M^+) \subseteq \tilde{C} \tag{2.2.}$$

As $V \subseteq \phi(\tilde{V})$, it suffices to prove the upper bound on growth of $\operatorname{Coeff} C$ for the subalgebra generated by $\phi(\tilde{V})$. We can also assume that the conformal algebra is smaller, in particular that it is generated by f_i's.

Consequently, we change notations and take C to be the conformal algebra generated by $f_1, \dots, f_n$. Put $V = \phi(\tilde{V})$ where $\tilde{V}$ is defined as in (2.2). We shall also use the explicit description of C_r's given by (2.1).

Put $N = \max_{j_1,j_2} N(f_{j_1}, f_{j_2})$ and $M_r^+ = rM^+, M_r^- = r(M^- - N)$. Recall that $M^- \le 0$. It is clear that as functions of r, M_r^+ increases and M_r^- decreases.

We shall study the growth of $\dim V^r$ via the growth of certain subspaces of $\tilde{V}^r$ in $(\tilde{C}, ⓪)$. The inductive statement claims that for every element of $V^1 + \cdots + V^r$ one can choose a preimage in the subspace $\tilde{V}_r$ of $\tilde{C}$ spanned by ht^k such that $h \in C_r$ and $M_r^- \le k \le M_r^+$. (Simultaneously, we will prove that $\tilde{V}^r \subseteq \tilde{V}_r$).

When passing from r to $r+1$, the statement automatically holds for elements of $V^i, 1 \le i \le r$ as $C_r \subseteq C_{r+1}$ and $(M_r^-, M_r^+) \subseteq (M_{r+1}^-, M_{r+1}^+)$. Therefore, to prove the claim one needs only to provide a procedure for choosing the preimage of an element in $V^{r+1} = V^r V^1$. By induction we can consider a larger space, namely $\phi(\tilde{V}_r)V$ and work only with the basis elements of $\phi(\tilde{V}_r)$ and V.

Let $gt^{k_1} \in \tilde{V}_r$ where $M_r^- \le k \le M_r^+$ and $g \in C_r$ is a product of f_j's. Consider $f_{j_2}t^{k^2} \in \tilde{V}$. We have from (1.5)

$$gt^{k_1} ⓪ f_{j_2}t^{k_2} = \sum_{j \ge 0} \binom{k_1}{j} (g ⓙ f_{j_2}) t^{k_1+k_2-j}.$$

Notice that $j \le N$ by lemma 2.2. Therefore $\phi(gt^{k_1} ⓪ f_{j_2}t^{k_2})$ lies in

$$\begin{aligned} &\phi(\operatorname{Span}_{\mathbb{C}}(g ⓙ f_{j_2}t^{k_1+k_2-j} \mid M_r^- \le k_1 \le M_r^+, M_1^- \le k_2 \le M_1^+, j \le N)) \subseteq \\ \subseteq &\phi(\operatorname{Span}_{\mathbb{C}}(ht^k \mid h \in C_{r+1}, M_r^- + M^- - N \le k \le M_r^+ + M^+) = \\ = &\phi(\operatorname{Span}_{\mathbb{C}}(ht^k \mid h \in C_{r+1}, M_{r+1}^- \le k \le M_{r+1}^+) = \tilde{V}_r. \end{aligned} \tag{2.3}$$

The immediate consequence is that $\dim V^1 + \cdots + V^{r+1}$ is bounded by the dimension of the subspace of $\tilde{V}_{r+1}$ given in (2.3). It does not exceed $\dim \tilde{V}_{r+1} = (\operatorname{rk} C_{r+1})(M_{r+1}^+ - M_{r+1}^-)$. We conclude that

$$\dim V^1 + \cdots + V^r \le (M_1^+ - M_1^- + N')r \cdot \operatorname{rk} C_r$$

and the theorem follows from the definition of GKdim. □

Remark 2.5. A similar result was proven for Lie conformal algebras of growth 0 in [DK, Lemma 5.2].

Example 2.6. Let C be a conformal algebra which is a torsion $\mathbb{C}[\partial]$-module. Clearly $\operatorname{GKdim} C = 0$, furthermore, it follows from (C2) that any finitely generated subalgebra of C is finite over $\mathbb{C}$. For any $f \in C$, let α_i be such that $(\Pi_{i=0}^{n}(\partial + \alpha_i))f = 0$. Put $f_j = (\Pi_{i=j}^{n}(\partial + \alpha_i))f$. It is clear from (1.6) that the coefficients of f_1 are proportional to either $f_1(0)$ or $f_1(-1)$, namely for all n, $\alpha_0 f_1(n) = n f_1(n-1)$. By induction, the coefficients of f are linear combinations of $f_j(0)$ and $f_j(-1)$, $1 \leq j \leq n-1$. Therefore, the coefficient algebra of a finitely generated subalgebra of C spanned over $\mathbb{C}$ by some g_i's is spanned over $\mathbb{C}$ by $\{(g_i)_j(0), (g_i)_j(-1)\}$ and is finite over $\mathbb{C}$. It follows that $\operatorname{GKdim} \operatorname{Coeff} C = 0$. This example shows that the inequality in Theorem 2.4 is sometimes strict.

3. Unital Conformal Algebras.

We begin by introducing the analogue of identity in ordinary associative algebras.

Definition 3.1. An element $e \in C$ is called a conformal identity if for every $f \in C$ $e \, (0) \, f = f$ and $N(e, e) = 1$. A conformal algebra containing such an element is called unital.

Remark 3.2. Conformal identity is not unique. Consider, for example, the current algebra over a unital algebra A that contains a nilpotent element r, $r^2 = 0$. Then both $f_{\mathbf{1}}$ and $f_{\mathbf{1}+r} = f_{\mathbf{1}} - \partial f_r$ are conformal identities in $\operatorname{Cur}(A)$ (here $\mathbf{1}$ is the identity in A). An obvious generalization of this example will be used in the proof of Theorem 4.2.

As we will demonstrate immediately, the Weyl algebra fits the definition. Furthemore, in a sense it is the only such simple algebra of growth 1 (see Theorem 4.6 for details).

It is one of the nice properties of Lie conformal algebras that every torsion element is central ([DK, Prop. 3.1]). In general, this is not true for associative algebras; however, for unital algebras an even stronger result holds.

Lemma 3.3. *A unital conformal algebra C is always torsion-free.*

Proof. The statement can be deduced from [DK]; however, an easier proof is possible. Let $f \in C$ be a torsion element. Since $\mathbb{C}$ is algebraically closed, we can always assume without loss of generality that $(\partial + \alpha)f = 0$ for some $\alpha \in \mathbb{C}$. Clearly for all $n > 0, g \in C$, $(\partial + \alpha)(g \, (n) \, f) = -ng \, (n-1) \, f$. If $g \, (0) \, f \neq 0$, $g \, (n) \, f \neq 0$ for all $n \geq 0$ contradicting locality of g and f. Thus, C is the left annihilator of f which contradicts the definition of e. $\square$

It is an open question whether every torsion-free conformal algebra can be embedded into a unital conformal algebra. This question can be reformulated in terms

of the coefficient algebras as well (see Corollary to Proposition 3.5). We remark that such an embedding automatically exists if the conformal algebra possesses a finite faithful representation as it implies that the algebra can be embedded into CEnd_N [K1, ch.2].

We now turn to the description of unital associative conformal algebras. The following fact is well-known (see [Ro] for a fuller exposition). The proof is elementary; however, some of the explicit calculations it contains will be needed below.

Lemma 3.4. *Let A be an associative algebra, δ a locally nilpotent derivation of A. Then the set $F = \{f_a = \sum at^n z^{-n-1}, a \in A\}$ of series over a localized Ore extension $A[t, t^{-1}; \delta]$ of A is a conformal algebra with the operations $f_a \, (m) \, f_b = f_{(-1)^m a\delta^m(b)}$.*

Proof. Let a, b be two elements of A. Consider the series $f_a = \sum at^n z^{-n-1}$ and $f_b = \sum bt^n z^{-n-1}$. The k-th coefficient of their m-th product is $\sum_s (-1)^s \binom{m}{s} at^{m-s} bt^{k+s}$. Since $t^n b = \sum_i (-1)^i \binom{n}{i} \delta^i(b) t^{n-i}$, we have

$$(f_a \, (m) \, f_b)(k) = \sum_{s,i} (-1)^{s+i} \binom{m-s}{i} \binom{m}{s} a\delta^i(b) t^{m+k-i}.$$

If $i \neq m$, $\sum_s (-1)^{s+i} \binom{m-s}{i} \binom{m}{s} = 0$, thus $(f_a \, (m) \, f_b)(k) = (-1)^m a\delta^m(b)t^k$. Hence,

$$f_a \, (m) \, f_b = f_{(-1)^m a\delta^m(b)}. \tag{3.1}$$

It follows immediately that F is closed with respect to taking conformal products. Locality follows from the nilpotency of δ. □

Conformal algebras described above are called *differential algebras.*

If one takes $A = C[x]$ and $\delta = \partial/\partial_x$, the resulting conformal algebra built as above is precisely the Weyl algebra $\mathfrak{W}$.

Notice that for the differential algebras constructed in Lemma 3.4, $\mathrm{GKdim}\, C = \mathrm{GKdim}\, \mathrm{Coeff}\, C - 1$. Indeed, the only osbtruction to the equality in the proof of Theorem 2.4 are non-zero products of generators with a zero zeroeth coefficient. However, if a formal distribution f_a has $f_a(0) = 0$, then it is zero itself.

We will now establish the converse of Lemma 3.4.

It should be noted first that for a conformal identity e, $e(0)$ is a left but not necessarily right identity of $\mathrm{Coeff}\, C$. Clearly, if the left annihilator of an associative algebra is 0, the left identity is also a right identity. We need to introduce a similar condition for conformal algebras. We call the set

$$L(C) = \{f \in C \mid f \, (n) \, g = 0 \text{ for all } g \in C, n \in \mathbb{Z}\}$$

the left annihilator of C. Notice that due to (1.8) and (1.9) $L(C)$ is a nilpotent ideal of C.

Proposition 3.5. *Let C be a unital conformal algebra with the conformal identity e. Let C have a trivial left annihilator. Then $\operatorname{Coeff} C = A[t, t^{-1}; \delta]$ where $\operatorname{GKdim} A = \operatorname{GKdim} C$ and δ is a locally nilpotent derivation of A. If C is finitely generated, then so is A.*

Proof. Notice first that $(e \, (1) \, e)(0) = e(1)e(0) - e(0)e(1) = 0$. Thus $e(0)$ is an identity in the subalgebra of $\operatorname{Coeff} C$ generated by $e(0)$ and $e(1)$. Later we will show that $e(0)$ is an identity in $\operatorname{Coeff} C$.

Consider, as before, the algebra $\operatorname{Coeff} C$ as an image of

$$\phi : C[t, t^{-1}] \to C[t, t^{-1}]/(\partial + \partial_t)C[t, t^{-1}].$$

Let A be the image of C in $\operatorname{Coeff} C$. For the conformal identity e, (1.5) implies

$$et^{n+1} = et^n \, (0) \, et - \sum_{j=1}^{n} \binom{n}{j} (e \, (j) \, e) t^{n+1-j} = et^n \, (0) \, et.$$

By induction, $et^n = (et)^n$, $n > 0$. Since $et^n \, (0) \, et^{-n} = e \, (0) \, e$, we have

$$e = \sum_{n \in \mathbb{Z}} t^n z^{-n-1}. \tag{3.2}$$

(With a slight abuse of notations, we denote $\phi(et)$ by t.) In these notations $t^0 = e(0)$.

It will follow that $\operatorname{Coeff} C$ is generated by the algebra of 0-th coefficients $A = \phi(C)$, t and t^{-1} if one can choose a set of generators $\{g_i\}$ of C such that $g_i \, (0) \, e = g_i$. Notice that for such elements $g(0)e(0) = g(0)$, thus it will follow that $e(0)$ is indeed the identity of $\operatorname{Coeff} C$. In general, in terms of coefficients $g_i(n) = g_i(0)t^n$ for all n. We will now show how every element of C can be expressed via the elements of this particular type.

Assume at first that $e(0)$ is an identity in $\operatorname{Coeff} C$.

Consider now an arbitrary element $f = \sum f(n) z^{-n-1}$. Put $\widehat{f(n)} = f(n)t^{-n}$. From the locality condition (1.4) in Lemma 1.1, we see that

$$\sum_{j=0}^{N(f,e)+1} (-1)^j \binom{N(f,e)+1}{j} \widehat{f(n-j)} = 0 \quad \text{for all } n > N(f,e), \tag{3.3}$$

as $f(n-j)e(j) = \widehat{f(n-j)} t^{n-j} \cdot t^j$. It follows that for any n, $\widehat{f(n)}$ is determined by a polynomial of degree $N(f,e)$ over $\operatorname{Coeff} C$, namely $\widehat{f(n)} = f_0 + f_1 n + \cdots +$

$f_{N(f,e)}n^{N(f,e)}$. Indeed, if $N(f,e)=0$, then $\widehat{f(n)} = f\widehat{(n-1)}$ and the result is clear. In the case of arbitrary $N(f,e)$, the locality condition (3.3) can be rewritten as

$$\sum_{j=0}^{N(f,e)} (-1)^j \binom{N(f,e)}{j} (f\widehat{(n-j)} - f\widehat{(n-j+1)}) = 0 \tag{3.4}$$

follows by induction.

Thus, we can rewrite f:

$$f = \sum (f_0 + f_1 n + \cdots + f_{N(f,e)} n^{N(f,e)}) t^n z^{-n-1}.$$

If $N(f,e)=0$, the element $f = \sum f_0 t^n z^{-n-1}$ already has the form we need. In the general case, consider the product $f \circledN e$, $N = N(f,e)$. Its coefficients are

$$(f \circledN e)(n) = \sum (-1)^j \binom{N}{j} (f_0 + f_1(N-j) \cdots + f_N(N-j)^N) t^{N-j} t^{n+j}. \tag{3.5}$$

In the expression above the coefficient at $f_k t^{N+n}$ is

$$\sum (-1)^j \binom{N}{j} (N-j)^k = \left\{ \begin{array}{ll} 0, & k < N \\ N!, & k = N \end{array} \right. .$$

Therefore, $f \circledN e = \sum (N! f_N t^N) t^n z^{-n-1}$ and has the form we need. It immediately follows that $\partial^N (f \circledN e) = \sum (-1)^N n(n-1) \cdots (n-N) N! f_N t^N z^{-n-1}$ and the n-th coefficient of

$$f - (-1)^N \frac{1}{N!} \partial^N (f \circledN e)$$

is a polynomial in n of degree $N-1$ with constant coefficient. It is easy to see that its degree of locality with e is also at most $N-1$. Hence, by repeating the process described above we can express f as a linear combiantion of monomials of elements in the desired form. By taking as f each generator of C we obtain a set of generators $\{g_i\}$ such that for all i and n, $g_i(n) = g_i(0)t^n$. Moreover, if C is finitely generated, this set is also finite.

We now return to the general case (i.e. $e(0)$ is not necessarily an identity). Consider a map $\Xi : C \to \operatorname{Coeff} C[[z, z^{-1}]]$ where $\Xi(f) = \sum f(n)e(0)z^{-n-1}$. Denote the image of Ξ by C'. Clearly $\Xi : C \to C'$ is a linear map. Moreover, since $e(0)$ is a left identity, $f(k)g(m)e(0) = (f(k)e(0)g(m)e(0)$ and (1.2) implies $\Xi(f \circledn g) = \Xi(f) \circledn \Xi(g)$. Hence C' is a unital conformal algebra of formal distributions over $(\operatorname{Coeff} C)e(0)$. We do not claim here that $\operatorname{Coeff} C' = (\operatorname{Coeff} C)e(0)$. However, the above construction of a particular generating set $\{g_i\}$ of a unital conformal algebra

does not use the universality of a coefficient algebra. Hence, C' has such a system of generators: $\{g_i' = \sum g_i'(0)t^n z^{-n-1}\}$.

Notice now that $\Xi : C \to C'$ is a bijection. Indeed, if $\Xi(f) = 0$, it follows that since $f(k)e(0) = 0$ for all k, $f(k)g(m) = 0$ for all $g \in C, m, k \in \mathbb{Z}$. Thus $f \, (n) \, g = 0$ and $f \in L(C)$, a contradiction.

Put $g_i = \Xi^{-1}(g_i')$, g_i' as above. Notice that g_i's are generators of C. Then $g_i(n)e(0) = g_i'(0)t^n = g_i(0)e(0)t^n = g_i(0)t^n = (g_i \, (0) \, e)(n)$. Thus $\Xi(g_i) = g_i' = g_i \, (0) \, e \in C$ and C' is a subalgebra of C. Moreover, for every $f \in C, n \in \mathbb{Z}_{\geq 0}$, (1.2) implies that $(g_i - \Xi(g_i)) \, (n) \, f = 0$. Hence $\Xi(g_i) = g_i$ and $C' = C$.

We conclude that C possesses a system of generators $\{g_i\}$ such that $g(n) = g(0)t^n$. In particular this implies that $\operatorname{Coeff} C = A[t, t^{-1}]$ with identity $e(0)$.

Define now a derivation $\tilde{\delta}(f) = f \, (0) \, et - et \, (0) \, f$ in $(\tilde{C}, \, (0))$. Then, on generators

$$\tilde{\delta}(f) = f \, (0) \, et - (e \, (0) \, ft + e \, (1) \, f) = -e \, (1) \, f. \tag{3.6}$$

Next, we introduce a derivation $\delta(\phi(f)) = \phi(\tilde{\delta}(f)$ on A. In this setup, $\operatorname{Coeff} C$ is an Ore extension of A. Moreover,

$$\begin{aligned} \tilde{\delta}^n(f) =& e \, (1) \, (e \, (1) \, \tilde{\delta}^{n-2} f) = \\ =& (e \, (0) \, e) \, (2) \, \tilde{\delta}^{n-2} f - (e \, (1) \, e) \, (1) \, \tilde{\delta}^{n-2} f = e \, (2) \, \tilde{\delta}^{n-2}(f) \end{aligned}$$

and by induction, $\tilde{\delta}^n f = (-1)^n e \, (n) \, f$. Therefore, δ is locally nilpotent.

It follows from the calculation in Lemma 3.4 that if C is finitely generated, then so are A and $\operatorname{Coeff} C$. In particular, the generators of $\operatorname{Coeff} C$ are the zeroeth coefficients of the generators of C, their δ-derivatives, and t. For unital algebras, $\phi(f_a) = 0$ if and only if $f_a = 0$. Thus, if in the proof of Theorem 2.4 one chooses the generating subspace of $\operatorname{Coeff} C$ as described above, all inequalities turn into equalities, hence $\operatorname{GKdim} \operatorname{Coeff} C = \operatorname{GKdim} C + 1$.

By [KL, 3.5 and 4.9] $\operatorname{GKdim} A = \operatorname{GKdim} A[t, t^{-1}; \delta] - 1 = \operatorname{GKdim} C$. $\square$

Corollary. *An associative conformal algebra C, such that $L(C) = 0$, can be embedded into a unital algebra if and only if it its coefficient algebra can be embedded into a localized Ore extension defined by a locally nilpotent derivative.*

Nonisomorphic unital conformal algebras C_1 and C_2 yield nonisomorphic pairs (A_1, δ_1) and (A_2, δ_2). Indeed, even if $A_1 \cong A_2$, then $\delta_1 \neq \delta_2$, otherwise by Lemma 3.4 and, in particular, (3.1), there exists a canonical isomorphism between C_1 and C_2. Clearly non-isomorphic A_1 and A_2 give rise to non-isomorphic C_1 and C_2; however, pairs (A, δ_1), (A, δ_2) with $\delta_1 \neq \delta_2$ may lead to the same conformal algebra. Consider, for example, $A = \mathcal{M}\mathrm{at}_n(\mathbb{C}[x])$ with $\delta_1 = \partial/\partial x$, and $\delta_2 = \partial/\partial x + \operatorname{ad} a$ where $a \in A$ is a nilpotent matrix (the construction of an isomorphism between (A, δ_1) and (A, δ_2) follows the proof of Theorem 4.6).

4. Unital Conformal Algebras of Small Growth.

The most immediate implication of Proposition 3.6 is that any classification of unital conformal algebras is much easier than the general case because one needs only to describe the associative algebra A and its derivation δ. Here we will do just that for conformal algebras of growth 0 and 1 over $\mathbb{C}$ (Actually, any algebraically closed field of characteristic 0). For the rest of this chapter, C always denotes a unital associative conformal algebra with the zero left annihilator, and A and δ are the resulting associative algebra and its locally nilpotent derivation.

Lemma 4.1. *C is simple if and only if A contains no δ-stable ideals.*

Proof. If I is a δ-stable ideal of A, then $I[t,t^{-1}]$ is an ideal of $\operatorname{Coeff} C$ and $J = \{f_a \mid a \in I\}$ an ideal of C by calculations in Lemma 3.4.

Conversely, let let J be a an ideal of C and put $I = \phi(J)$, the set of all 0-th coefficients of J. Clearly $a \in I$ if and only if $f_a \in J$. Since for any $a \in I$, $b \in A$, $f_{ab} = f_a \, ⓪ \, f_b \in J$, I is an ideal of A. Moreover, since $f_{\delta^n(a)} = (-1)^n (e \, ⓝ \, f_a) \in J$, I is a δ-stable ideal of A.

It remains to show that I is non-empty. Since C is a coefficient algebra, for any $f \in C$, there exists a collection $\{a_i\} \subset A$ such that $f = \sum_{i,k} c_{ik} \partial^k f_{a_i}$. Therefore, for every m,

$$
\begin{aligned}
f(m) =& \sum_{i,k} c_{ik} (\partial^k f_{a_i})(m) = \sum_{i,k} (-1)^m \frac{m!}{(m-k)!} c_{ik} f_{a_i}(m-k) = \\
=& \sum_k \Big(\sum_i c_{ik} a_i \Big) \Big((-1)^m \frac{m!}{(m-k)!} t^{m-k} \Big)
\end{aligned}
\tag{4.1}
$$

(if $m \geq 0$, we sum only over $k \leq m$). It follows that if $f(m) = 0$ for all $m \geq 0$, then $\sum_i c_{ik} a_i = 0$ for all k implying $f(m) = 0$ for $m < 0$.

Let now f be an element of J, and $m \geq 0$ minimal such that $f(m) \neq 0$. Then $(f \, ⓜ \, e)(0) \neq 0$. $\square$

Corollary. *Let C be an assocItive current algebra $Cur(A)$ (not necessarily unital) where A contains more than 1 element. Then C is simple iff A is simple.*

Proof. The proof is the same as the proof of the Lemma; it is only required that A contain a such that $f \, ⓜ \, f_a \neq 0$, f and m as above. If $f(m)A = 0$, then there exists $b \in A$, $bA = 0$. It follows that A is not simple. $\square$

Theorem 4.2. *If C is a simple unital associative conformal algebra and a finite $\mathbb{C}[\partial]$-module, then it is a current algebra over the ring of matrices over $\mathbb{C}$.*

Proof. By Proposition 3.5 $\operatorname{GKdim} A = 0$ and Lemma 4.1 A is δ-simple. It follows immediately from [Bl] that $A = \mathcal{M}\mathrm{at}_n(\mathbb{C})$. It is well-known that δ is necessarily an

inner derivation $\operatorname{ad} a$(see, e.g. [H]). It is easy to show that there exists a nilpotent $r \in A$ such that $\delta = \operatorname{ad} r$ (r is the nilpotent part of a).

Clearly, the ring of skew-polynomials $A[t, t^{-1}; \delta]$ is isomorphic to the ring of polynomials $A[s, s^{-1}]$ (send $t - r$ to s). To prove that the corresponding differential algebras are isomorphic as well, it suffices to show that the series $e' = \sum (t - r)^n z^{-n-1}$ belongs to the differential algebra over $A[t, t^{-1}; \delta]$. Indeed, in this case the conformal algebra generated by e' and $\left(\sum a t^n z^{-n-1}\right) (0)\, e'$ is canonically isomorphic to $\mathrm{Cur}(A)$.

Let m be the degree of nilpotency of r. Since $\delta(r) = 0$, t and r commute, therefore,

$$e' = \sum_{k=0}^{m-1} \frac{1}{k!} \partial^k \left(\sum r^k t^n z^{-n-1} \right) \in \text{diff. algebra over } A[t, t^{-1}; \delta].$$

□

The classification of simple associative algebras of finite type (i.e. those finite as $\mathbb{C}[\partial]$-modules) [K2, 4.4] implies that all simple algebras of finite type are unital. However, so far there exists no direct way to demonstrate that a simple algebra of finite type can be adjoined a conformal identity.

The next task is to classify simple unital conformal algebras of growth 1. Of great help is the classification of finitely generated associative algebras of growth 1 developed in [SW] and [SSW]. We will use the following result of the former:

Theorem [SW]**.** *A finitely generated prime algebra A, GKdim $A = 1$ is necessarily a finite module over its center which is also finitely generated.*

Remark 4.3. The immediate corollary of this classification is an otherwise non-obvious fact that there exist no finitely generated simple associative current conformal algebras of linear growth. Indeed, let $C = \mathrm{Cur}(A)$ be such an algebra. A is simple as well by corollary to Lemma 4.1 and $\operatorname{GKdim} A = 1$. Hence, its center is a finitely generated simple algebra of GKdim 1, i.e. a field of transcedental degree 1. Such can not be finitely generated over $\mathbb{C}$ as an algebra.

Assume now that C (hence A) is finitely generated. The follwong lemma is well-known.

Lemma 4.4. [Po] *Let A be a differentially simple algebra as above. Then A is prime.*

We begin the classification of unital conformal algebras of growth 1 (i.e. the classification of pairs (A,δ) up to the isomorphism of resulting conformal algebras) with the commutative case.

Lemma 4.5. *If A is commutative (not necessarily finitely generated), then either $\delta \equiv 0$ and A is a field of transdendental degree one or $A = \mathbb{C}[x]$ and $\delta = \partial/\partial x$ for some $x \in A$.*

Proof. Consider an element whose derivative is 0. It generates a δ-stable ideal of A, hence it is invertible and the set of all such elements forms a subfield of A.

It follows that the set of algebraic elements of A whose derivative is 0 forms an algebraic subfield of A, i.e. is $\mathbb{C}$. Let $x \in A$ be transcendental and n minimal such that $\delta^n(x) = 0$. Two cases can occur: either $\delta^{n-1}(x) \in \mathbb{C}$ (we can always assume that it is 1) or $\delta^{n-1}(x)$ is transcendental. Hence, the set of all transcendental elements of C splits into two, each corresponding to one of the cases above. We will show that these cases do not occur simultaneously (i.e. either one of the subsets is necessarily empty). Pick an arbitrary element from each set, respectively x and y. Without loss of generality we may assume that $\delta(x) = 1, \delta(y) = 0$. These two elements are algebraically dependent since $\operatorname{GKdim} A = 1$ but in every statement of dependence of y over $\mathbb{C}[x]$ the degree in x can be lowered by the application of δ making y algebraic over $\mathbb{C}$. Thus these cases do not occur simultaneously.

If we assume that no transcendental element has a non-zero derivative, i.e. $\delta \equiv 0$, A is a field of transcendental degree 1 over $\mathbb{C}$. Moreover, if at least one transcendental element has a zero derivative, we can apply the argument from the last paragraph: if there exists an element with a non-zero derivative, then there exist x and y be such that $\delta(x) = y$, $\delta(y) = 0$. But y is invertible and $\delta(y^{-1}x) = 1$. We conclude that if one transcendental element has a zero derivative, then every transcendental element does and A is a field.

The remaining case follows from a lemma in [W]; however, in the current framework one can provide an elemetary proof. If there exists x such that $\delta(x) = 1$, consider an arbitrary $y \in A$ with $\delta(y) = x$. We have $\delta(y - x^2/2) = 0$ which by the previous discussion means that $y = x^2 + a, a \in \mathbb{C}$ (i.e. y is an "antiderivative" of x). For an arbitrary $y \in A$ we conclude by induction on n such that $\delta^n(y) = 0$, that $y \in \mathbb{C}[x]$. $\square$

We already know that for a simple unital conformal algebra C, $\operatorname{Coeff} C = A[t, t^{-1}; \delta]$ where δ is a locally nilpotent derivative and A is a prime algebra of GKdim 1. It follows from [SW] that A is a finite module over its center $Z(A)$ which is also of GKdim 1. Notice that δ can always be restricted to $Z(A)$ which is δ-simple as well. Indeed, assume the contrary, i.e. the existence of $a \in Z(A)$ that, together with its derivatives, generates a δ-stable ideal of $Z(A)$. Since δ is locally nilpotent, without loss of generality we may assume that $\delta(a) = 0$ and that $\langle a \rangle$ is an ideal of $Z(A)$. But by δ-simplicity of A, there exists $b \in A$, $ab = 1$, thus for any $x \in A$, $xb - bx = ba(xb - bx) = b(x - x) = 0$. Hence, a is invertible in $Z(A)$,

a contradiction. Thus we can apply lemma 4.5 to the classification of the general case.

Theorem 4.6. *Let C be a finitely generated simple unital associative conformal algebra of linear growth. Then C is isomorphic to the differential algebra of formal distributions over $\mathcal{M}at_n(W_t)$.*

Proof. From the Proposition 3.5 and Lemma 4.4, $\operatorname{Coeff} C = A[t, t^{-1}; \delta]$, $Z(A) = \mathbb{C}[x]$, $\delta|_{\mathbb{C}[x]} = \partial/\partial x$.

Consider subalgebra $A_0 = \ker \delta$. We begin by demonstrating that A_0 generates A over $Z(A)$. More precisely, every $a \in A$ is of the form $\sum_1^n \frac{x^i}{i!} a_i$, $a_i \in A_0$ where n is such that $\delta^n(a) = 0, \delta^{n-1}(a) \neq 0$. Indeed, with the induction assumption $\delta(a) = \sum_1^{n-1} \frac{x^i}{i!} b_i$, $b_i \in A_0$, consider $a_0 = a - \sum_1^{n-1} \frac{x^{i+1}}{(i+1)!} b_i$. Since $\delta(a_0) = 0$, we see that a is also a polynomial in x over A_0. Moreover, this polynomial form is unique for any $a \in A$, i.e. if $\sum_1^n x^i a_i = 0$, applying a necessary number of derivatives shows that the coefficient at the highest power is 0.

Fix a subset $\{a_i\}$ of A_0 that generates A over $Z(A)$. Any product of elements of A_0 is a linear combination $\sum p_i(x) a_i$ over $Z(A)$ with the derivative $\sum (\partial p_i(x)/\partial x) a_i = 0$. This implies $A_0 = \operatorname{Span}_{\mathbb{C}}(a_i)$ is finite dimensional.

Clearly any ideal of A_0 can be lifted to A, thus A_0 is prime as well and therefore simple over $\mathbb{C}$ ([R, 2.1.15]). Hence $A = \mathcal{M}\mathrm{at}_n(\mathbb{C}[x])$ and $\delta = \partial/\partial x$. It follows that $A[t, t^{-1}; \delta]$ is a matrix ring over $\mathbb{C}[x][t, t^{-1}; \partial/\partial x] = W_t$. $\square$

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